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The stochastic, damped KdV equation

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Abstract. We apply singular perturbation theory to the study of a damped KdV soliton under the influence of space- and time-dependent external noise. We find that asymptotically the shape of the averaged solution approaches that of a Gaussian packet, whose amplitude decays and width grows in the same way as in the case of purely time-dependent noise.

1. Introduction

In this paper we will consider the effects of external noise on the damped Korteweg-deVries (KdV) equation in the form

$$u_t + 6uu_x + u_{xxx} = \varepsilon\zeta(x, t)R[u] - \varepsilon\gamma u. \tag{1}$$

This problem has been studied by Wadati and Akutsu [1], based on earlier work of Wadati [2] on the stochastic KdV equation without damping, where the external noise was purely time-dependent and $R[u]=1$. Recently, plasma physicists have examined the propagation of an ion-acoustic soliton in the presence of noise. We begin in section 2 by presenting a possible derivation of the stochastic KdV equation from the plasma fluid equations. This will be followed by a discussion of the theoretical and experimental results of the above-mentioned works.

Equation (1) can be viewed as a perturbation of the KdV equation, which is a well known integrable evolution equation. In the late 1970s several researchers investigated such perturbations of the KdV equation, but did not apply these results to the stochastic KdV equation given in (1). The methods used were based on the inverse scattering transform (IST), and involved the study of the effects of the perturbation on the so-called scattering data. In section 3 and appendix 1 we sketch a more direct approach to investigating perturbed KdV equations. In the sections to follow we apply the results of this method to the study of the stochastic KdV equation for the cases of space-independent and space-dependent noise, as well as the cases of damping and no damping. These results are then compared with those which have been found previously.

2. Motivation and derivation

In 1966 Washimi and Taniuti [3] derived the KdV equation as the equation governing the propagation of small-amplitude ion-acoustic waves in one dimension, using the reductive perturbation technique. Fluctuations in the dynamical variables can lead to perturbations of the KdV equation. In this section we will employ the reductive

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perturbation method to obtain a stochastic KdV equation, where the stochastic term can arise from the fluctuations induced by external forces.

The starting point of this analysis is the set of plasma fluid equations for a collisionless, magnetic field-free plasma of cold ions and warm electrons ($T_i \leq T_e$) [4]. Under the assumption of isothermal electrons with negligible mass, satisfying the equation of state $P_e = n_e k T_e$, the electron fluid equation reduces to the equation

$$kT_e \nabla n_e = e n_e \nabla \Phi \tag{2}$$

which can be solved to give

$$n_e = n_0 \exp\left(\frac{e\Phi}{kT_e}\right). \tag{3}$$

The other equations needed are the ion fluid equation of motion

$$\frac{\partial}{\partial t} \boldsymbol{v}_i + (\boldsymbol{v}_i \cdot \nabla)\boldsymbol{v}_i = -\frac{e}{M} \nabla \Phi + \boldsymbol{F}(\boldsymbol{x}, t) \tag{4}$$

where $\boldsymbol{F} = -(\nabla p/\rho) + (1/\rho)\boldsymbol{F}_{ext}$ is the contribution from external and pressure forces, which will be the source of noise for our problem. We also have Poisson's equation

$$-\nabla^2 \Phi = 4\pi e(n_i - n_e) \tag{5}$$

and the equation of continuity for ions

$$\frac{\partial}{\partial t} n_i + \nabla \cdot (n_i \boldsymbol{v}_i) = 0. \tag{6}$$

In these equations k is Boltzmann's constant, T_e is the electron temperature, n_e (n_i) is the electron (ion) number density, $\boldsymbol{v}_i$ is the ion velocity, and Φ is the electric potential.

Equations (3)-(6) can be cast into a dimensionless form by the transformations

$$\begin{array}{llll} x' = x/\lambda_D & t' = c_s t/\lambda_D & \boldsymbol{v}' = \boldsymbol{v}/c_s & n' = n_i/n_0 \\ \phi = e\Phi/kT_e & n_e' = n_e/n_0 & \boldsymbol{F}' = \boldsymbol{F}\lambda_D/c_s^2 & \end{array} \tag{7}$$

where $\lambda = (kT_e/4\pi n_0 e^2)^{1/2}$ is the Debye length and $c_s = (kT_e/M)^{1/2}$ is the ion-acoustic velocity. The dimensionless plasma fluid equations are

$$\nabla n_e = n_e \nabla \phi \Rightarrow n_e = e^{\phi} \tag{8}$$

$$\boldsymbol{v}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla \phi + \boldsymbol{F} \tag{9}$$

$$n - n_e = -\nabla^2 \phi \tag{10}$$

$$n_t + \nabla \cdot (n\boldsymbol{v}) = 0 \tag{11}$$

after dropping the primes. As we are only interested in one dimension, we define the component of the ion velocity by u and rewrite the fluid equations as:

$$u_t + u u_x = -\phi_x + F \tag{12}$$

$$n - e^{\phi} = -\phi_{xx} \tag{13}$$

$$n_t + (nu)_x = 0. \tag{14}$$

We can now apply the reductive perturbation technique to this system. Following Washimi and Taniuti [3], we define the scaled variables by

$$\xi = \varepsilon^{1/2}(x - Vt) \qquad \tau = \varepsilon^{3/2} t \tag{15}$$

where V is the velocity of propagation in the x direction. Since $\{n=1,\ u=0,\ \phi=0\}$ is a solution of the system (12)-(14) when $F=0$, we can expand about this solution using the series

$$n=1+\varepsilon n_1+\varepsilon^2 n_2+\dots \qquad u=0+\varepsilon u_1+\varepsilon^2 u_2+\dots$$
$$\phi=0+\varepsilon\phi_1+\varepsilon^2\phi_2+\dots \qquad F=0+\varepsilon^{3/2}F_1+\dots. \tag{16}$$

Applying the transformations (15) and expansions (16) to the system (12)-(14), one obtains, to lowest order in ε,

$$V^2=1 \qquad n=u_1/V=\phi_1\equiv\psi. \tag{17}$$

To the next order, after some algebra, one finds the equation

$$\psi_\tau+\psi\psi_\xi+\tfrac{1}{2}\psi_{\xi\xi\xi}=F_1. \tag{18}$$

The stochastic KdV equation can be obtained when we assume that the external force $F(x,t)$ is Gaussian white noise for small ε. In the following we will define Gaussian noise by the averages [2]

$$\langle F_1(x_1,t_1)\dots F_1(x_n,t_n)\rangle_s=\begin{cases}\Sigma\Pi\langle F_1(x_i,t_i)F_1(x_j,t_j)\rangle_s & n \text{ even}\\ 0 & n \text{ odd}\end{cases} \tag{19a}$$

and by white noise we will mean

$$\langle F_1(x,t)F_1(x',t')\rangle_s=N\delta(x-x')\delta(t-t'). \tag{19b}$$

$\Sigma\Pi$ means that we multiply $n/2$ products $\langle F(x_i,t_i), F(x_j,t_j)\rangle_s$, and sum over the $(n-1)!!$ different combinations.

Before leaving this derivation we call attention to related works, which have involved water waves. In 1972 Yajimi [5] studied the effect of irregularities in the bottom surface on the progpagation of small-amplitude water waves. In his derivation of the governing equation he expanded the relevant dependent variables in a Fourier series and averaged over the oscillating terms. The resulting equation was in the form of the KdV equation plus a complicated perturbing term which consists of sums of Fourier integrals.

In 1976 Kawahara studied the effect of random inhomogeneities on the propagation of water waves. In his derivation he used the method of smoothing [6] to arrive at a perturbed KdV equation, which involved integrals containing particular correlations of the random function. In the special case in which these correlations are localised, such as Gaussian white noise, the perturbed equation reduces to a form of the KdV-Burgers equation.

In 1975 Wadati [2] studied the stochastic KdV equation of the form

$$u_t-6uu_x+u_{xxx}=\zeta(t) \tag{20}$$

where $\zeta(t)$ is the time-dependent external noise. He showed that this equation was connected to a KdV equation through the Galilean transformation

$$u(x,t)=U(X,t)+W(t) \qquad W(t)=\int_0^t \mathrm{d}t'\zeta(t') \qquad X=x+6\int_0^t \mathrm{d}t'W(t') \tag{21}$$

leading to

$$U_t-6UU_X+U_{XXX}=0. \tag{22}$$

Thus, for a fixed $\zeta(t)$, the equation is exactly solvable. He computed the ensemble average of $u(x, t)$ by expanding the sech^2 form of a one-soliton solution, in order to evaluate the statistical averages over sums of products of the stochastic variable, to obtain

$$\langle u(x, t)\rangle_s = 8\eta^2 \sum_{n=1}^{\infty} (-1)^n n \exp(an + bn^2) \tag{23}$$

where

$$a = 2\eta(x - x_0 - 4\eta^2 t) \qquad b = 48\varepsilon\eta^2 t^2 \qquad \langle\zeta(t)\zeta(t')\rangle_s = 2\varepsilon\delta(t - t'). \tag{24}$$

This form of $\langle u(x, t)\rangle_s$ is the solution of the diffusion-like equation

$$\frac{\partial}{\partial b}\langle u(x, t)\rangle_s = \frac{\partial^2}{\partial a^2}\langle u(x, t)\rangle_s \tag{25}$$

where b acts as a *time*. Noting that for $b = 0$

$$\langle u(x, t)\rangle_s|_{b=0} = -2\eta^2 \mathrm{sech}^2(a/2) \tag{26}$$

this can be solved using the usual Fourier transform technique [2] to obtain

$$\langle u(x, t)\rangle_s = -8\eta^2 \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{\pi k}{\sinh(\pi k)} \mathrm{e}^{-bk^2} \mathrm{e}^{\mathrm{i}ak}. \tag{27}$$

The asymptotic behaviour of the solution in (25) can be found for large and small times. For large t, Wadati obtained

$$\langle u(x, t)\rangle_s \sim -\frac{\eta}{\sqrt{3\pi\varepsilon}} t^{-3/2} \exp\left(-\frac{(x - x_0 - 4\eta^2 t)^2}{48\varepsilon t^3}\right). \tag{28}$$

Thus, we see that the soliton amplitude decreases as $t^{-3/2}$, while the width grows as $t^{3/2}$. Furthermore, under the effect of the random perturbation the soliton changes its basic shape, becoming a Gaussian packet. In later sections of this paper we will use the same approach to compute ensemble averages.

Wadati and Akutsu [1] extended this study to include the effects of damping and included N-soliton solutions. They again assumed that the noise was only time dependent. In the case of damping they relied on the perturbation results of Karpman and Maslov for the damped KdV equation [15]. The general large-time behaviour for the cases of damping and non-damping were given in the forms

$$\begin{aligned} &\text{No damping} \qquad \langle u(x, t)\rangle_s \sim \text{constant} \times t^{-3/2} \exp\left(-\text{constant} \times \frac{x^2}{t^3}\right) \\ &\text{Damping} \qquad \langle u(x, t)\rangle_s \sim \text{constant} \times t^{-1/2} \mathrm{e}^{-\gamma t} \exp\left(-\text{constant} \times \frac{x^2}{t}\right). \end{aligned} \tag{29}$$

The width is narrower and the amplitude is smaller in the damped case.

In 1986 Blaszak extended the work of Wadati and Akutsu to the stochastic KdV hierarchy and the stochastic MKdV equation [7, 8]. The form of the stochastic MKdV equation, which he treated, is given by

$$u_t + 6u^2 u_x + u_{xxx} = \varepsilon\eta(t) \tag{30}$$

while that for the stochastic KdV hierarchy with damping can be written as

$$u_t + \phi^n u_x = \varepsilon\eta(t) + \gamma u \tag{31}$$

where ϕ is the recursion operator [7] generating the hierarchy. This operator is defined as

$$\phi = D^2 + 4u + 2u_x D^{-1} \tag{32}$$

where $D = d/dx$ and $D^{-1} = \int^x dx$. We note that for $n = 1$ the stochastic KdV results.

Blaszak uses his results on Lax pairs for equations in soliton hierarchies [9] combined with the perturbation method of Karpman and Maslov to obtain asymptotic results in time, when the external noise is Gaussian. For $n = 1$ he obtains the same solution that Wadati and Akutsu had reported. We see that Blaszak has also assumed that the noise is purely time dependent.

Over the past twenty years there has been much experimental research into solitons in plasma physics. Recently a group of experimentalists studied the propagation of ion-acoustic solitons in a *noisy* plasma [10, 20]. They found that such solitons are damped, and that their velocity decreases. They found that the amplitude decayed as $t^{-2/3}$, whereas the study of Wadati, without damping, gave $t^{-3/2}$, and that of Wadati and Akutsu gave $t^{-1/2}$. The experimentalists had compared their results with Wadati's result for no damping. They had suggested that the agreement was good, as there was considerable room for error in their system. We note that the experimental value actually lies in between the two results.

Our goal in this paper is to study equation (1) as a perturbed KdV equation, using a direct approach. In the next section we present this direct approach. Having done this, we will first turn to the case of purely time-dependent Gaussian white noise and verify that we obtain the same results as discussed in this section. This analysis will then be followed by an analysis of the general form in equation (1) using the same methods.

3. Perturbations of the KdV equation

Since the mid 1970s several papers have been produced to describe the effects of perturbations on soliton solutions of integrable nonlinear evolution equations [11, 12]. Kaup [13] suggested using the IST to study singular perturbations of these equations. Kaup and Newell (KN) [14] had applied this method to the KdV equation, as well as other perturbed equations. Around the same time Karpman and Maslov (KM) had also studied perturbations of the KdV equation [15], using inverse spectral methods. In general, it was found that the effects of small perturbations could lead to a change in the shape and position, or phase shift, of the initial soliton.

We now summarise the results of a direct approach to solving the perturbed equation [16]

$$u_t + 6uu_x + u_{xxx} = \varepsilon F[u] \tag{33}$$

subject to the initial condition

$$u(x, 0) = 2\eta^2 \operatorname{sech}^2 \eta x. \tag{34}$$

For small perturbations, we expect that the solution will remain close to the soliton solution for some time. Therefore, the solution we seek will be roughly a soliton with a slowly changing shape and location plus a correction. In order to insure that we are close to the soliton solution, we assume an asymptotic expansion of the form

$$u(x, t) = u_0(x, t) + \varepsilon u_1(x, t) + \dots . \tag{35}$$

To account for the slowly changing soliton parameters, we take

$$u_0(x, t) = 2\eta^2 \operatorname{sech}^2 \eta\left(x + \frac{1}{\varepsilon} x_0 + x_1\right) \tag{36}$$

and we define the two time scales, $T = t$ and $\tau = \varepsilon t$. We further assume that the soliton parameters η, x_0 and x_1 depend only on the slow scale τ.

Introducing the expansions (35) and (36) and the two time scales into equation (33), we obtain an expansion of (33) in powers of ε. Setting the coefficients of each order of ε to zero, we obtain a system of equations to be solved for u_n. The lowest-order equation is the KdV equation, which will be satisfied if

$$x_{0\tau} = 4\eta^2. \tag{37}$$

The first-order equation then becomes

$$\mathscr{L}u_1 = -4\eta\eta_\tau v - 2\eta\eta_\tau \phi v_\phi + 2\eta^3 x_{1\tau} v_\phi + F[u_0] \tag{38}$$

where $\mathscr{L}$ is the linearised KdV operator

$$\mathscr{L} \equiv \partial_T - 4\eta^3 \partial_\phi + 6\eta \partial_\phi u_0 + \eta^3 \partial_\phi^3. \tag{39}$$

The problem is now to invert this operator. In appendix 1 we discuss the details of this inversion for the general problem

$$\mathscr{L}u_1 = \mathscr{F}. \tag{40}$$

Very simply, we expand u_1 in an appropriate basis $\{\Phi^A, \Phi_1^A, \Lambda_1^A\}$ as

$$u_1 = \int_{-\infty}^{\infty} f(\lambda, t)\Phi^A(x, t; \lambda)\, \mathrm{d}\lambda + f_1(t)\Phi_1^A(x, t) + g_1(t)\Lambda_1^A(x, t). \tag{41}$$

In the one-soliton case the expansion coefficients are shown to take the form

$$f(\lambda, t) = \int_0^t \mathrm{d}t' \frac{\langle \mathscr{F} | \Phi \rangle}{2\pi \mathrm{i} \lambda a^2(\lambda)} \exp[8\mathrm{i}\lambda^3(t - t')] \tag{42}$$

$$g_1(t) = -2\mathrm{i}\eta \int_0^t \mathrm{d}t' \langle \mathscr{F} | \Phi_1 \rangle \exp[8\eta^3(t - t')] \tag{43}$$

$$f_1(t) = -2\mathrm{i}\eta \int_0^t \mathrm{d}t' \langle \mathscr{F} | \Lambda_1 \rangle \exp[8\eta^3(t - t')] + 96\,\mathrm{i}\eta^3 \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \langle \mathscr{F} | \Phi_1 \rangle \exp[8\eta^3(t - t'')] \tag{44}$$

where the inner product is defined by

$$\langle f(x) | g(x) \rangle \equiv \int_{-\infty}^{\infty} f(x) g(x)\, \mathrm{d}x. \tag{45}$$

Using the basis states for a one-soliton solution, we can rewrite the last two terms in (41),

$$B \equiv f_1 \Phi_1^A + g_1 \Lambda_1^A \tag{46}$$

as

$$B = \tilde{g}_1[\operatorname{sech}^2 \phi + \tfrac{1}{2}\phi(\operatorname{sech}^2 \phi)_\phi] + \tilde{h}_1(\operatorname{sech}^2 \phi)_\phi \tag{47}$$

where the new coefficients are given by

$$\tilde{g}_1 \equiv \int_0^t \mathrm{d}t' \langle \mathscr{F} | \mathrm{sech}^2 \phi \rangle \tag{48}$$

$$\tilde{h}_1 \equiv -\frac{1}{2} \int_0^t \mathrm{d}t' \langle \mathscr{F} | [\phi + 8\eta^3 (t - t')] \mathrm{sech}^2 \phi + \tanh \phi \rangle. \tag{49}$$

We note that for $\mathscr{F}$ independent of time these coefficients will grow in time, unless we impose the secularity conditions

$$\langle \mathscr{F} | \mathrm{sech}^2 \phi \rangle = 0 \tag{50}$$

$$\langle \mathscr{F} | \phi \, \mathrm{sech}^2 \phi + \tanh \phi \rangle = 0. \tag{51}$$

Applying these conditions to equation (38), we obtain the slow time dependence of the soliton parameters [16]

$$x_{0\tau} = -4\eta^2 \tag{52}$$

$$\eta_\tau = \frac{1}{4\eta} \int_{-\infty}^{\infty} F[u_0] \, \mathrm{sech}^2 \phi \, \mathrm{d}\phi \tag{53}$$

$$x_{1\tau} = \frac{1}{4\eta^3} \int_{-\infty}^{\infty} F[u_0] (\phi \, \mathrm{sech}^2 \phi + \tanh \phi + \tanh^2 \phi) \, \mathrm{d}\phi. \tag{54}$$

The first equation determines the change in the soliton amplitude and width $(1/\eta)$. The second of these equations gives the leading-order velocity, while the last equation will give the correction to the velocity of the soliton.

We have found the effects of the perturbation on the soliton parameters in equations (52)-(54). From the analysis we are left with the first-order correction to the solution as

$$u_1 = \int_{-\infty}^{\infty} f(\lambda, t) \Phi^A(x, t; \lambda) \, \mathrm{d}\lambda. \tag{55}$$

In the following section we will use these results to investigate the effects of external noise on the soliton shape and location.

4. Solitons in time-dependent noise

We first turn to the equation

$$u_t + 6uu_x + u_{xxx} = \varepsilon \zeta(t) - \varepsilon \gamma u \tag{56}$$

which is the case of time-dependent Gaussian white noise with damping. We begin by letting

$$u(x, t) = \hat{u}(x, t) + \varepsilon h(t).$$

Equation (56) then becomes

$$\hat{u}_t + 6\hat{u}\hat{u}_x + \hat{u}_{xxx} = \varepsilon \zeta(t) - \varepsilon \gamma \hat{u} - \varepsilon h_t - \varepsilon^2 \gamma h - \varepsilon h \hat{u}_x - 6\varepsilon h \hat{u}_x. \tag{57}$$

Using the usual perturbation expansions

$$\tau = \varepsilon t \qquad \hat{u} = u_0 + \varepsilon \hat{u}_1 + \ldots \tag{58}$$

the zeroth-order and first-order equations which we must solve become

$$u_{0_t} - x_{0_\tau} u_{0_\tau} + 6\eta u_0 u_{0_\phi} + \eta^3 u_{0_{\phi\phi\phi}} = 0 \tag{59}$$

$$\mathscr{L}(\hat{u}_1) = \mathscr{S}_1 - 2\eta^2\gamma \operatorname{sech}^2\phi - (h_\tau - \zeta + \varepsilon\gamma h) \tag{60}$$

where

$$\mathscr{S}_1 \equiv 2\eta^3 x_{1_\tau} v_\phi - 4\eta\eta_\tau v - 2\eta\eta_\tau \phi v_\phi \qquad v = \operatorname{sech}^2\phi. \tag{61}$$

Note that we have kept the purely time-dependent terms $h_\tau - \zeta + \varepsilon\gamma h$ in the first-order driving terms. These will be used to determine $h(t)$.

We have to avoid certain divergences caused by these extra terms in equation (59). During the inversion of the first-order equation, we run into a term of the form

$$\int_0^t \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\langle\Phi\rangle(h_\tau - \zeta + \varepsilon\gamma h)}{2\pi\mathrm{i}\lambda a^2(\lambda)\eta} \exp[8\mathrm{i}\lambda^3(t-t')]\Phi^A(\phi, t; \lambda). \tag{62}$$

Carrying out this integral leads to a divergent term in the expression for $\hat{u}_1$, since we have a $\delta(\lambda)/\lambda$ from the inner product under the integral

$$\langle\Phi\rangle \equiv \int_{-\infty}^{\infty} \mathrm{d}\phi\Phi(\phi, t; \lambda) = \left(\frac{\eta}{2}(\eta^2 - \lambda^2)\delta(\lambda)\right)\frac{\exp(-8\mathrm{i}\lambda\eta^2 t)}{(\mathrm{i}\lambda - \eta)^2}. \tag{63}$$

In order to avoid this, we set

$$h_\tau + \varepsilon\gamma h = \zeta \tag{64}$$

which leads to

$$h(t) = \int_0^t \zeta(s)\exp[\varepsilon\gamma(s-t)]\,\mathrm{d}s. \tag{65}$$

(Note that we have assumed that the perturbation is turned on at $t = 0$; thus, $h(0) = 0$.)

Now we can proceed as usual. Employing the secularity conditions (50) and (51), we find

$$\eta_\tau = -\tfrac{2}{3}\gamma\eta \qquad x_{1_\tau} = -\frac{\gamma}{3\eta} + 6h. \tag{66a}$$

Solving for $\hat{u}_1$, we obtain the same form as the first-order solution to the purely damped KdV equation [16]

$$\hat{u}_1 = \int_{-\infty}^{\infty} \frac{\gamma(\eta^2+\lambda^2)[\exp(8\mathrm{i}\lambda^3 t) - \exp(-8\mathrm{i}\lambda\eta^2 t)]}{12\lambda(\mathrm{i}\lambda+\eta)^3(\mathrm{i}\lambda-\eta)\sinh(\pi\lambda/\eta)}\Phi^A(\phi, t; \lambda)\,\mathrm{d}\lambda. \tag{66b}$$

Denoting this by $u_{1\mathrm{d}}$, the full first-order solution to equation (56) is given by

$$u_1(x, t) = u_{1\mathrm{d}}(x, t) + h(t). \tag{67}$$

We can obtain results from averaging u_0 in a fixed frame, as Wadati and Akutsu had done in [1, 2]. We will generalise their procedure by writing u_0 as

$$u_0(x, t) = (A_0 + \varepsilon W)\operatorname{sech}^2(\phi_0 + \varepsilon\eta_0 Q) \tag{68}$$

where A_0 and ϕ_0 are the coherent parts of the amplitude and the phase, and εW, $\varepsilon\eta_0 Q$ are the fluctuating parts. For the case above, we have

$$W = 0 \qquad \eta_0 Q = 6\int_0^t h(s)\,\mathrm{d}s. \tag{69}$$

However, in the next section we will need the more general form in equation (68), though the computations for averaging will essentially be the same.

Following Wadati, we expand the $\operatorname{sech}^2 \phi$ in (68) as

$$u_0 = 4(2\eta_0^2 + \varepsilon W) \sum_{m=1}^{\infty} (-1)^{m+1} m \exp(2m\phi_0) \exp(2\varepsilon m \eta_0 Q). \tag{70}$$

The types of averages which will occur in averaging this expression, are derived in appendix 2 as

$$\langle e^{cV} \rangle_s = \exp(\tfrac{1}{2}c^2 \langle V^2 \rangle_s) \tag{71}$$

$$\langle W e^{cV} \rangle_s = c \langle VW \rangle_s \exp(\tfrac{1}{2}c^2 \langle V^2 \rangle_s). \tag{72}$$

Evaluating the average of u_0 in (68), using these relations, gives

$$\langle u_0 \rangle_s = 4(2\eta_0^2 + 2\varepsilon\eta_0 \langle QW \rangle_s \partial_a) \sum_{m=1}^{\infty} (-1)^{m+1} m \exp(am + bm^2) \tag{73}$$

where a and b are defined by

$$a = 2\phi_0 \tag{74}$$

$$b = 2\varepsilon^2 \eta_0^2 \langle Q^2 \rangle_s. \tag{75}$$

Noting that

$$w(a, b) \equiv 4 \sum_{m=1}^{\infty} (-1)^{m+1} \exp(am + bm^2) \tag{76}$$

satisfies a diffusion equation,

$$w_b - w_{aa} = 0 \tag{77}$$

with the *initial condition*

$$w(b = 0) = \operatorname{sech}^2(a/2) \tag{78}$$

the average of the solution in equation (68) can be written as

$$\langle u_0 \rangle_s = (2\eta_0^2 + 2\varepsilon\eta_0 \langle QW \rangle_s \partial_a) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\pi k}{\sinh(\pi k)} \exp(-bk^2 + iak). \tag{79}$$

The behaviour of the summation for $b > 1$ and $b < 1$ for $W = 0$ was given by Wadati. For $b < 1$, $\exp(-bk^2)$ can be expanded, leading to the expression

$$\langle u_0(x, t) \rangle_s = 2\eta_0^2 \sum_{n=0}^{\infty} \frac{b^n}{n!} \frac{\partial^{2n}}{\partial a^{2n}} \operatorname{sech}^2\left(\frac{a}{2}\right) = 2\eta_0^2 \operatorname{sech}^2\left(\frac{a+b}{2}\right). \tag{80}$$

We note that Wadati and Akutsu did not perform the summation, which in our later results can yield an interpretation as to the initial effects of the noise on the soliton.

For $b > 1$, we can use the expansion [2]

$$\frac{\pi k}{\sin(\pi k)} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2^{2n} - 2) B_n \pi^{2n} k^{2n}}{(2n)!}$$

where B_n is the nth Bernoulli number. Substituting this into equation (79) gives

$$\langle u_0 \rangle_s \sim \frac{2}{\sqrt{\pi}} (2\eta_0^2 + 2\varepsilon\eta_0 \langle QW \rangle_s \partial_a) \left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2^{2n} - 2) B_n \pi^{2n} k^{2n}}{(2n)!}\right) \frac{\exp(-a^2/4b)}{\sqrt{b}}. \tag{81}$$

What these expressions tell us about the physics of the model can only be found out after computing the ensemble averages $\langle WQ\rangle_s$ and $\langle Q^2\rangle_s$. As before, we expect to be able to say something about the asymptotic behaviour of the solution for large times, or for small times.

For the case at hand, where W and Q are given by equations (69), a and b are now given by

$$a=2\phi_0 \tag{82}$$

$$b=2\eta^2\left\langle\left(6\varepsilon\int_0^t h(s)\,\mathrm{d}s\right)^2\right\rangle_s . \tag{83}$$

We need to compute the average in the definition of b, equation (75)

$$\langle Q^2\rangle_s=36\varepsilon^2\int_0^t \mathrm{d}t_1\int_0^t \mathrm{d}t_2\langle h(t_1)h(t_2)\rangle_s . \tag{84}$$

To do this, we first need to compute the correlation $\langle h(t_1)h(t_2)\rangle_s$. Using (40), we find

$$\begin{aligned}\langle h(t_1)h(t_2)\rangle_s &= N\exp[-\varepsilon\gamma(t_1+t_2)]\int_0^{t_1}\mathrm{d}s\int_0^{t_2}\mathrm{d}s'\exp[\varepsilon\gamma(s+s')]\delta(s-s')\\ &=\frac{N}{2\varepsilon\gamma}\exp[-\varepsilon\gamma(t_1+t_2)]\{\exp[2\varepsilon\gamma\min(t_1,t_2)]-1\}\\ &=\frac{N}{2\varepsilon\gamma}\{\exp(-\varepsilon\gamma|t_1-t_2|)-\exp[-\varepsilon\gamma(t_1+t_2)]\}.\end{aligned} \tag{85}$$

Inserting this into equation (59), we obtain the desired result as

$$\begin{aligned}\langle Q^2(t)\rangle_s &=\frac{18\varepsilon N}{\gamma}\int_0^t \mathrm{d}t_1\left(\int_0^{t_1}\mathrm{d}t_2\exp[-\varepsilon\gamma(t_1+t_2)]\right.\\ &\quad\left.+\int_{t_1}^t \mathrm{d}t_2\exp[-\varepsilon\gamma(t_2-t_1)]-\int_0^t \mathrm{d}t_2\exp[-\varepsilon\gamma(t_1+t_2)]\right)\\ &=\frac{18N}{\varepsilon\gamma^3}[2\varepsilon\gamma t-3+4\exp(-\varepsilon\gamma t)-\exp(-2\varepsilon\gamma t)].\end{aligned} \tag{86}$$

Therefore, we have found b as a function of t

$$b=\frac{36\eta^2 N}{\varepsilon\gamma^2}[2\varepsilon\gamma t-3+4\exp(-\varepsilon\gamma t)-\exp(-2\varepsilon\gamma t)]. \tag{87}$$

For large t and $\gamma\neq 0$, we have

$$b\sim\frac{72\eta^2 N}{\gamma}t. \tag{88}$$

Therefore, we can study the large-time behaviour of the average solution by using equation (81), for $b>1$. The result is given by

$$\langle u_0(x,t)\rangle_s\sim\left(\frac{\eta_0}{3}\sqrt{\frac{2\gamma}{\pi N}}\right)t^{-1/2}\exp(-\tfrac{2}{3}\varepsilon\gamma t)\exp[-(\gamma/72Nt\eta^2)\phi_d^2] \tag{89}$$

for $t\gg\gamma/72\eta^2 N$. Therefore, we see that the averaged solution behaves as a damped Gaussian packet, whose width broadens as $t^{1/2}$ and whose amplitude decays as $t^{-1/2}$, as found by Blaszak [7]. Actually, though Wadati has reported the argument of the damping factor as $-\varepsilon\gamma t$, in fact a check of their computations yields the same results as above, which is not surprising.

For small times

$$b \approx 24\varepsilon^2\gamma\eta^2 N t^3. \tag{90}$$

Thus, for $t \ll (24\eta^2 N\varepsilon^2\gamma)^{-1/3}$ we obtain the result from equation (80)

$$\begin{aligned}\langle u_0(x,t)\rangle_s &\simeq 2\eta^2 \operatorname{sech}^2\left(\frac{a+b}{2}\right)\\ &= 2\eta_0^2 \exp(-\tfrac{4}{3}\varepsilon\gamma t)\operatorname{sech}^2(\phi_d + 12\eta^2 N\gamma\varepsilon^2 t^3).\end{aligned} \tag{91}$$

We see that on the average the initial effects are to cause the amplitude to decay, the width to broaden, and the soliton to slow down.

For the case of no damping, where $\gamma = 0$, we can take the limit $\gamma \to 0$ in equation (87) for b. Doing this, we find

$$b \approx 24 N\eta^2\varepsilon^2 t^3. \tag{92}$$

Using (81) we find for the large-time behaviour to be

$$\langle u_0(x,t)\rangle_s \sim \frac{\eta}{\sqrt{3\pi N/2}}\, t^{-3/2} \exp\left(-\frac{(x-\bar{x})^2}{24Nt^3}\right) \tag{93}$$

for $t \gg (24N\eta^2\varepsilon^2)^{-1/3}$. This result indicates that the soliton asymptotically approaches the form of a Gaussian, whose amplitude decays as $t^{-3/2}$ and whose width grows as $t^{3/2}$. Again, we have found agreement with Wadati for the case without damping [5].

For small t, we use equation (80) to find

$$\langle u_0(x,t)\rangle_s = 2\eta^2 \operatorname{sech}^2[\tfrac{1}{2}(\phi_0 + 24N\eta^2\varepsilon^2 t^3)]. \tag{94}$$

Here we see that for small times the soliton again begins to slow down, since the velocity can be written as

$$v \equiv \frac{\partial}{\partial t}\left(\frac{a+b}{2}\right) = 4\eta^2 - 96N\eta^2\varepsilon^2 t^2. \tag{95}$$

Thus, we have shown that through the application of the proposed perturbation method, combined with ensemble averaging in a fixed frame, we are able to reproduce the results of Blaszak [7] and Wadati and Akutsu [1, 2].

5. Solitons in time- and space-dependent noise

In this section we investigate the results of the perturbation method on the damped, stochastic KdV equations for external noise depending on both x and t. The form of the equations which we will use is given by

$$u_t + 6uu_x + u_{xxx} = \varepsilon\zeta(x,t)R[u] - \varepsilon\gamma u. \tag{96}$$

We will use the general perturbation analysis to obtain the first-order solution of the problem. We will evaluate the ensemble averages in a fixed frame in the same manner as we have done in the last section.

Note that the form of the stochastic term involves a function of u, which is different from the stochastic terms used in the last sections. Previously we had considered the case of $R[u]=1$; however, there are some divergence problems with this type of noise. The new form, (96), is an obvious generalisation of the equation treated in section 2, and does not possess the divergence problems.

After developing the first-order solution and averages for a general $R[u]$, we will turn to the cases $R[u]=u$ and $R[u]=u_x$ in (96). These cases could correspond to dissipation and velocity fluctuations, respectively. For example, if we transform the KdV equation in equation (18) back to the original coordinates ($F_1\equiv 0$), we have the form

$$u_t+c_s u_x+uu_x+\tfrac{1}{2}\lambda_D^2 c_s u_{xxx}=0 \tag{97}$$

where $c_s=(kT_e/M)^{1/2}$ is the ion-acoustic velocity, u the ion speed, and $\lambda_D=(kT_e/4\pi n_0 e^2)$. For regions where u_{xxx} is small, fluctuations in T_e will cause fluctuations in c_s, which in turn will lead to a stochastic term with $R[u]=u_x$.

More generally, we could return to the plasma fluid equations in dimensionless form in (12)-(14). Assuming that fluctuations can result in any of the physical processes, which they represent, we could write

$$u_t+uu_x+\phi_x=A \tag{98}$$

$$n-e^{\phi}+\phi_{xx}=B \tag{99}$$

$$n_t+(nu)_x=C \tag{100}$$

where A, B and C contain the fluctuations from each equation. Carrying out the analysis in section 1, a perturbed KdV equation results:

$$\psi_\tau+\psi\psi_\xi+\tfrac{1}{2}\psi_{\xi\xi\xi}=A+C+B_\xi. \tag{101}$$

Thus, fluctuations in u, n, or ϕ might be introduced, determining the forms of A, B and C.

We now return to the perturbation analysis of equation (96). Employing the usual perturbation analysis, which we used in earlier sections, we find for the amplitude

$$A_\tau=4\eta\eta_\tau=\langle\zeta R|\operatorname{sech}^2\phi\rangle-\tfrac{4}{3}\gamma A \tag{102}$$

$$x_{0_\tau}=4\eta^2=2A \tag{103}$$

$$x_{1_\tau}=\frac{1}{4\eta^2}\langle\zeta R|\phi\operatorname{sech}^2\phi+\tanh\phi+\tanh^2\phi\rangle-\frac{\gamma}{3\eta}. \tag{104}$$

Integrating equation (102), we have

$$\eta^2=\frac{\varepsilon}{2}\int_0^t\langle\zeta(s)R|\operatorname{sech}^2\phi\rangle\exp[\tfrac{4}{3}\varepsilon\gamma(s-t)]\,ds. \tag{105}$$

We now assume that $R[u]$ depends on η according to

$$R[u_0]\equiv\eta^p\hat{R}[u_0]. \tag{106}$$

Equation (105) is then an integral equation for η. By iterating we can, in principle, solve for η to any order in ε. To first order in ε we find

$$\eta^2\simeq\eta_0^2\left(1+\frac{\varepsilon}{2}\eta_0^{p-2}\int_0^t\langle\zeta(s)\hat{R}|\operatorname{sech}^2\phi\rangle\exp[\tfrac{4}{3}\varepsilon\gamma(s-t)]\,ds\right) \tag{107}$$

$$\eta\simeq\eta_0\left(1+\frac{\varepsilon}{4}\eta_0^{p-2}\int_0^t\langle\zeta(s)\hat{R}|\operatorname{sech}^2\phi\rangle\exp[\tfrac{4}{3}\varepsilon\gamma(s-t)]\,ds\right). \tag{108}$$

We can use this information to finish integrating the conditions in equations (102)-(104). Thus, the other soliton parameters are given to first order as

$$x_0 = x_0(0) + 4\varepsilon_0^2\left(t + \frac{\varepsilon}{2}\,\eta_0^{p-2} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}s' \langle \zeta(s')\hat{R}|\mathrm{sech}^2\,\phi\rangle \exp[\tfrac{4}{3}\varepsilon\gamma(s'-s)]\right) \tag{109}$$

$$A \simeq A_0 + \varepsilon \int_0^t \langle \zeta(s)\hat{R}|\mathrm{sech}^2\,\phi\rangle \exp[\tfrac{4}{3}\varepsilon\gamma(s-t)]\,\mathrm{d}s \tag{110}$$

$$x_1 \simeq x_1(0) + \frac{\varepsilon\eta_0^p}{4\eta_0^3} \int_0^t \mathrm{d}s \langle \zeta(s)\hat{R}|\phi\,\mathrm{sech}^2\,\phi + \tanh\phi + \tanh^2\phi\rangle - \frac{\varepsilon\gamma}{3\eta_0}\,t. \tag{111}$$

Finally, we can write the soliton solution in the form

$$u_0(x,t) = (A_0 + \varepsilon W)\mathrm{sech}^2(\phi_0 + \varepsilon\eta_0 Q) \tag{112}$$

which is in the same form we have worked with previously. (See equation (68).) Here we have defined

$$\phi_0 = \eta_0\chi_0 = \eta_0\left(x - 4\eta_0^2 t + \frac{\varepsilon\gamma t}{3\eta_0} - \frac{1}{\varepsilon}\,x_0(0) - x_1(0)\right) \tag{113}$$

$$W = \eta_0^p \int_0^t \langle \zeta(s)\hat{R}|\mathrm{sech}^2\,\phi\rangle \exp[\tfrac{4}{3}\varepsilon\gamma(s-t)]\,\mathrm{d}s \tag{114}$$

$$Q = -\frac{\eta_0^{p-3}}{4} \int_0^t \mathrm{d}s \langle \zeta(s)\hat{R}|\phi\,\mathrm{sech}^2\,\phi + \tanh\phi + \tanh^2\phi\rangle - 2\int_0^t \mathrm{d}s\, W(s) + \frac{1}{4\eta_0^2}\,\chi_0 W(t). \tag{115}$$

All that remains is to compute the ensemble averages $\langle QW\rangle_s$ and $\langle Q^2\rangle_s$. In this general formalism we can take a few more steps before specifying the form of $R[u]$. Namely, we can compute the averages $\langle QW\rangle_s$ and $\langle Q^2\rangle_s$, which are presented in appendix 3. The results are

$$\begin{aligned}\langle QW\rangle_s = &-\frac{3N\eta_0^{2p-3}}{16\varepsilon\gamma}\,G_1[1-\exp(-\tfrac{8}{3}\varepsilon\gamma t)] + \frac{3\eta_0^{2p-2}N}{32\varepsilon\gamma}\,\chi_0 G_2[1-\exp(-\tfrac{8}{3}\varepsilon\gamma t)]\\ &-\frac{9N\eta_0^{2p}}{16\varepsilon^2\gamma^2}\,G_2[1+\exp(-\tfrac{8}{3}\varepsilon\gamma t) - 2\exp(-\tfrac{4}{3}\varepsilon\gamma t)]\end{aligned} \tag{116}$$

and

$$\begin{aligned}\langle Q^2\rangle_s = &\frac{\eta_0^{2p-6}}{16}\,NG_3 t + \frac{27\eta_0^{2p}}{32\varepsilon^2\gamma^2}\,NG_2\left(\frac{8\varepsilon\gamma t}{3} + 4\exp(-\tfrac{4}{3}\varepsilon\gamma t) - 1 - \exp(-\tfrac{8}{3}\varepsilon\gamma t)\right)\\ &+\frac{3N\eta_0^{2p-4}}{128\varepsilon\gamma}\,\chi_0^2 G_2[1-\exp(-\tfrac{8}{3}\varepsilon\gamma t)]\\ &+\frac{9N\eta_0^{2p-3}}{16\varepsilon^2\gamma^2}\,G_1\left(\frac{4\varepsilon\gamma t}{3} + \exp(-\tfrac{4}{3}\varepsilon\gamma t) - 1\right) - \frac{3\eta_0^{2p-5}}{32\varepsilon\gamma}\,\chi_0 G_1[1-\exp(-\tfrac{4}{3}\varepsilon\gamma)]\\ &-\frac{9N\eta_0^{2p-2}}{32\varepsilon^2\gamma^2}\,G_2[1+\exp(-\tfrac{8}{3}\varepsilon\gamma t) - 2\exp(-\tfrac{4}{3}\varepsilon\gamma t)]\end{aligned} \tag{117}$$

where we have defined

$$G_1 \equiv \int_{-\infty}^{\infty} \mathrm{d}\phi\, \hat{R}^2[u_0](\phi\, \mathrm{sech}^4\, \phi + \mathrm{sech}^2\, \phi \tanh \phi + \mathrm{sech}^2\, \phi - \mathrm{sech}^4\, \phi) \tag{118}$$

$$G_2 \equiv \int_{-\infty}^{\infty} \mathrm{d}\phi\, \hat{R}^2[u_0]\, \mathrm{sech}^4\, \phi \tag{119}$$

$$G_3 \equiv \int_{-\infty}^{\infty} \mathrm{d}\phi\, \hat{R}^2[u_0](\phi\, \mathrm{sech}^2\, \phi + \tanh \phi + \tanh^2 \phi)^2. \tag{120}$$

We can study these averages to get information about $\langle u_0 \rangle_s$. In fact, to first order in ε we have that

$$\langle u_0 \rangle_s \simeq 4\eta_0^2 \sum_{m=1}^{\infty} (-1)^m m \exp(\hat{a}m + bm^2) \tag{121}$$

where

$$\hat{a} \equiv 2\phi_0 + (\varepsilon/\eta_0)\langle QW \rangle_s \equiv 2\hat{\phi}_0. \tag{122}$$

Using this new phase and the expansion in (81), we have for large b

$$\langle u_0 \rangle_s \sim \frac{4\eta_0^2}{\sqrt{\pi b}} \exp(-\hat{a}^2/4b). \tag{123}$$

We look now at the behaviour of these averages for large times and large x, such that χ_0 is fixed. Using $b = 2\varepsilon^2\eta_0^2\langle Q^2 \rangle_s$, we have

$$b \sim \frac{\eta_0^{2p-4}}{8} N\varepsilon^2 t\left(G_3 + 36\frac{\eta_0^6}{\varepsilon^2\gamma^2} G_2 + 12\frac{\eta_0^3}{\varepsilon\gamma} G_1\right) \tag{124}$$

and we also find

$$\langle QW \rangle_s \sim -\frac{3\eta_0^{2p-3}}{32\varepsilon^2\gamma^2}(2\varepsilon\gamma G_1 + 6\eta_0^3 G_2 - \eta_0\chi_0\varepsilon\gamma G_2). \tag{125}$$

Thus, from equation (123) we see that the behaviour of the averaged soliton solution yields a Gaussian wavepacket whose amplitude decays as $t^{-1/2}$ and whose width grows as $t^{1/2}$. Finally, we note that the effect of $\langle QW \rangle_s$ is to add a constant phase shift to the solution to first order in ε.

We can also obtain information about the case of no damping from the results in equations (116) and (117). Namely, we take the limit $\gamma \to 0$ in these equations and find

$$\langle QW \rangle_s \to \langle QW \rangle_{s,0} = -\frac{\eta_0^{2p-3} N}{4} t(2G_1 + 4\eta_0^2 G_2 t - \eta_0\chi_0 G_2) \tag{126}$$

$$\langle Q^2 \rangle_s \to \langle Q^2 \rangle_{s,0} = \frac{\eta_0^{2p-6}}{16} N(G_3 + 8\eta_0^3 t^2 G_1 - 2\eta_0\chi_0 G_1 t - 8\eta_0^4 G_2 t^2 + \tfrac{64}{3}\eta_0^2 G_2 t^3). \tag{127}$$

Now, the large-time behaviour is given by

$$\langle QW \rangle_{s,0} \sim -\eta_0^{2p} N G_2 t^2 \tag{128}$$

$$\langle Q^2 \rangle_{s,0} \sim \tfrac{4}{3}\eta_0^{2p} N G_2 t^3. \tag{129}$$

From the asymptotic expansion in equation (123) and the definition of b, this leads to a Gaussian wavepacket with an ampltiude decaying as $t^{-3/2}$ and width growing as

$t^{3/2}$. Thus, we have found that the same power laws result in the phase- and time-dependent noise cases under the approximations made above, as we had obtained in the purely time-dependent cases.

The above results have a generic character for any $R[u]$ of the form $\eta_0^{2p}\hat{R}[u]$, since we made the major lowest-order approximation of the integral equation for η back in equation (105). In principle, we can go back and carry this out to higher order in ε and establish the range of validity of this approximation.

For completeness, we can evaluate the integrals for the two particular cases (i) $R[u]=u$ and (ii) $R[u]=u_x$.

For these cases we have

$$\begin{array}{llll} \text{(i)} & G_1=\frac{24}{105} & G_2=\frac{128}{35} & G_3=\frac{928}{315} \\ \text{(ii)} & G_1=-\frac{32}{45} & G_2=\frac{512}{315} & G_3=\frac{512}{105}. \end{array} \qquad (130)$$

6. Conclusion

In this paper we have studied the stochastic KdV equation with and without damping for the case of time- and space-dependent noise. We have computed the ensemble averages for these cases with respect to a fixed frame; i.e. the averaging was performed with fixed x coordinates. In general we have found that asymptotically in time the noise causes the shape of the averaged solution to change from the $\text{sech}^2\,\phi$ form to that of a Gaussian packet, whose amplitude decays and whose width grows in much the same way as in the case of noise which is purely time dependent.

The method employed to study this equation was based on the inversion of the linearised KdV equation to obtain the first-order correction to the solution of equation (1) about a single soliton. In the process it was found that the soliton parameters had to obey *secularity* conditions in order to prevent any growth of the solution in time. Using these conditions in the leading order solution, we found the asymptotic behaviour of the solution.

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Appendix 1. Inversion of the linearised KdV equation

The major step in solving equation (38) is the inversion of the operator $\mathscr{L}$ given in (39). This can be done by using the squared Schrödinger eigenfunctions, their derivatives and some of the formalism from the theory of the inverse scattering transform (IST) [12]. We will begin by discussing the relationship between the KdV equation, the associated 'Lax pair', and the linearised operator we are seeking to invert. Having established this relation, we will then proceed to discuss the inversion of the operator for the special case of $u_0=2\eta^2\,\text{sech}^2\,\eta(x-\bar{x})$.

It is well known that the KdV equation

$$q_t + 6qq_x + q_{xxx} = 0 \tag{A1.1}$$

is an integrability condition for the equations [11, 12]

$$v_{xx} + (\lambda^2 + q)v = 0 \tag{A1.2}$$

$$v_t + v_{xxx} + 3(q - \lambda^2)v_x = \gamma v. \tag{A1.3}$$

Namely, $v_{txx} = v_{xxt}$ provided q satisfies (A1.1) and $\lambda_t = 0$. In these equations λ is an eigenvalue, and the constant γ is determined from the assumed asymptotic behaviour of v in the regions where q vanishes. In particular, if we assume that $v \sim e^{i\lambda x}$ ($v \sim e^{-i\lambda x}$) as $x \to \infty$ ($x \to -\infty$), then $\gamma = -4i\lambda^3$ ($4i\lambda^3$). Keeping with standard notation [12], we will denote such solutions as ψ_2 (ϕ_2).

We now consider the function $f = \partial_x(v^2)$, and operate on it with $\mathscr{L}$ from (39). Using equations (A1.1) and (A1.2), we find that f satisfies the eigenvalue problem

$$\mathscr{L}f = 2\gamma f. \tag{A1.4}$$

The aim of the following analysis is to use these eigenfunctions as a basis in which to expand the solution of equation (38). The unknown expansion coefficients will then be determined using orthogonality relations between the basis set and their adjoints. The details of this analysis rests on the solution of the Schrödinger eigenvalue problem (A1.2).

In the Schrödinger eigenvalue problem there is a continuous spectrum for $\lambda^2 > 0$ and possible bound states for $\lambda^2 < 0$. The eigenstates for the continuous spectrum of $\mathscr{L}$ are easily found from these λ; however, the bound states $\partial_x v^2|_{\lambda_k}$ are not sufficient to complete the set of states [17]. We find the required states from the work of Sachs [18]. If $f(x)$ is continuous and L^1, and if q satisfies

$$\|q\|_{L_2^1} \equiv \int_{-\infty}^{\infty} (1 + x^2) q(x)\, dx < \infty \tag{A1.5}$$

then Sachs shows how $f(x)$ can be expanded in the eigenfunctions of (A1.4). Applying his expansion to the one-soliton case, $q = u_0$ in (A1.2), one can write, using some notation from Newell [19],

$$f(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi i\lambda a^2(\lambda)} \Phi^A(x,\lambda)\Phi(y,\lambda) f(y)\, dy\, d\lambda$$
$$+ \frac{ic_1}{2\alpha_1}\int_{-\infty}^{\infty} (\Phi_1^A(x)\Lambda_1(y) - \Lambda_1^A(x)\Phi_1(y)) f(y)\, dy \tag{A1.6}$$

where

$$\alpha_1 \equiv \frac{\psi_2(x,\lambda_1)}{\phi_2(x,\lambda_1)} \qquad c_1 \equiv \left(\frac{i}{2\alpha_1}\int_{-\infty}^{\infty} \Phi_1^A(x)\Lambda_1(x)\, dx\right)^{-1} \tag{A1.7}$$

$$a(\lambda) = \frac{\lambda - i\eta}{\lambda + i\eta} \qquad \lambda_1 = i\eta. \tag{A1.8}$$

The desired complete set for perturbations about a one-soliton solution is given by $\{\Phi^A, \Phi_j^A, \Lambda_j^A\}$. These basis states are related to the Schrödinger eigenfunctions by

$$\Phi^A(x, t; \lambda) \equiv \partial_x \psi_2^2 \qquad \Phi_1^A(x,t) \equiv \partial_x \psi_2^2|_{\lambda_1} \tag{A1.9}$$

$$\Lambda_1^A(x,t) \equiv \partial_\lambda \partial_x \psi_2^2|_{\lambda_1} - \frac{1}{\alpha_1^2}\partial_\lambda \partial_x \phi_2^2|_{\lambda_1}. \tag{A1.10}$$

The *complementary*, or *adjoint*, states to these are given by

$$\Phi(x, t; \lambda) \equiv \phi_2^2 \qquad \Phi_1(x, t) \equiv \phi_2^2|_{\lambda_1} \tag{A1.11}$$

$$\Lambda_1(x, t) \equiv \partial_\lambda \phi_2^2|_{\lambda_1} - \alpha_1^2 \partial_\lambda \phi_2^2|_{\lambda_1}. \tag{A1.12}$$

Before returning to the inversion of the linearised KdV equation, we need two properties of the basis states. First, the result of operating on the basis states with $\mathscr{L}$ yields [16]

$$\mathscr{L}\Phi^A = -8i\lambda^3\Phi^A \qquad \mathscr{L}\Phi_1^A = -8i\lambda_1^3\Phi_1^A \qquad \mathscr{L}\Lambda_1^A = -8i\lambda_1^3\Lambda_1^A - 48i\lambda_1^2\Phi_1^A. \tag{A1.13}$$

Second, we will need the inner products between the basis states and the adjoint states [16]

$$\langle \Phi^a(\lambda)|\Phi(\lambda')\rangle = 2\pi i\lambda a^2(\lambda)\delta(\lambda - \lambda') \tag{A1.14}$$

$$\langle \Lambda_1^A|\Phi_1\rangle = \langle \Phi_1^A|\Lambda_1\rangle = \frac{i}{2\eta} \tag{A1.15}$$

$$\langle \Lambda_1^A|\Lambda_1\rangle = 0 = \langle \Phi_1^A|\Phi_1\rangle. \tag{A1.16}$$

The inner product $\langle f(x)|g(x)\rangle$ is defined by

$$\langle f(x)|g(x)\rangle = \int_{-\infty}^{\infty} f(x)g(x)\, dx. \tag{A1.17}$$

We can now proceed with the inversion of the linearised KdV equation for perturbations about a one-soliton solution,

$$\mathscr{L}u_1 = \mathscr{F}. \tag{A1.18}$$

We assume that u_1 can be expanded in the complete set of states as

$$u_1 = \int_{-\infty}^{\infty} d\lambda f(\lambda, t)\Phi^A(x, t;, \lambda) + f_1(t)\Phi_1^A(x, t) + g_1(t)\Lambda_1^A(x, t). \tag{A1.19}$$

Inserting this expansion into (A1.18), we have

$$\mathscr{F} = \int_{-\infty}^{\infty} d\lambda\, (f_t - 8i\lambda^3 f)\Phi^A(\lambda) + (f_{1_t} - 8\eta^3 f_1 + 48\eta^2 g_1)\Phi_1^A + (g_{1_t} - 8\eta^3 g_1)\Lambda_1^A. \tag{A1.20}$$

In order to obtain the coefficients of Φ^A, Φ_1^A and Λ_1^A, we take inner products on both sides of (A1.20) with the adjoint states and use the orthogonality relations (A1.14)-(A1.16) to obtain

$$f_t - 8i\lambda^3 f = \frac{\langle \mathscr{F}|\Phi\rangle}{2\pi i\lambda a^2(\lambda)} \tag{A1.21}$$

$$f_{1_t} - 8\eta^3 f_1 + 48\eta^2 g_1 = -2i\eta\langle \mathscr{F}|\Lambda_1\rangle \tag{A1.22}$$

$$g_{1_t} - 8\eta^3 g_1 = -2i\eta\langle \mathscr{F}|\Phi_1\rangle. \tag{A1.23}$$

These first-order differential equations for the expansion coefficients can easily be solved to obtain

$$f(\lambda, t) = f(\lambda, 0)\exp(8i\lambda^3 t) + \int_0^t dt' \frac{\langle \mathscr{F}|\Phi\rangle}{2\pi i\lambda a^2(\lambda)} \exp[8i\lambda^3(t - t')] \tag{A1.24}$$

$$g_1(t)=g_1(0)\exp(8\eta^3t)-2i\eta\int_0^t dt'\langle\mathscr{F}|\Phi_1\rangle\exp[8\eta^3(t-t')] \tag{A1.25}$$

$$f_1(t)=f_1(0)\exp(8\eta^3t)-48\eta^2tg_1(0)\exp(8\eta^3t)-2i\eta\int_0^t dt'\langle\mathscr{F}|\Lambda_1\rangle\exp[8\eta^3(t-t')]$$
$$+96i\eta^3\int_0^t dt'\int_0^{t'} dt''\langle\mathscr{F}|\Phi_1\rangle\exp[8\eta^3(t-t'')]. \tag{A1.26}$$

In our analysis we have at $t=0$, $u=u_0$; i.e. $u_n=0$ for $n>1$. So, in equations (A1.24)–(A1.26) the initial values of the expansion coefficients are zero.

This completes the inversion of the linear operator. The specific forms for the basis states and the adjoints is found by solving (A1.2) and (A1.3) for $q=u_0$. We list these for reference as

$$\Phi(x,t;\lambda)=\frac{\exp(-2i\lambda\phi/\eta-8i\lambda\eta^2t)}{(i\lambda-\eta)^2}(\eta^2\tanh^2\phi+2i\lambda\eta\tanh\phi-\lambda^2) \tag{A1.27}$$

$$\Phi_1(x,t)=\tfrac{1}{4}\exp(8\eta^3t)\operatorname{sech}^2\phi \tag{A1.28}$$

$$\Lambda_1(x,t)=-\frac{i}{\eta}\exp(8\eta^3t)[(\phi+4\eta^3t)\operatorname{sech}^2\phi+\tanh\phi] \tag{A1.29}$$

$$\Phi^A(x,t;\lambda)=\frac{2\exp(2i\lambda\phi/\eta+8i\lambda\eta^2t)}{(i\lambda-\eta)^2}$$
$$\times[-\eta^3\tanh^3\phi+2i\lambda\eta^2\tanh^2\phi+(2\lambda^2\eta+\eta^3)\tanh\phi-i(\lambda^3+\lambda\eta^2)] \tag{A1.30}$$

$$\Phi_1^A=-\tfrac{1}{2}\eta\exp(-8\eta^3t)\operatorname{sech}^2\phi\tanh\phi \tag{A1.31}$$

$$\Lambda_1^A=2i\exp(-8\eta^3t)[\operatorname{sech}^2\phi-(\phi+4\eta^3t)\operatorname{sech}^2\phi\tanh\phi]. \tag{A1.32}$$

Appendix 2. Ensemble averages $\langle\exp cV\rangle_s$ and $\langle W\exp cV\rangle_s$

In section 4 we stated that the ensemble averages $\langle\exp cV\rangle_s$ and $\langle W\exp cV\rangle_s$ can be computed as

$$\langle\exp cV\rangle_s=\exp(\tfrac{1}{2}c^2\langle V^2\rangle_s) \tag{A2.1}$$

$$\langle W\exp cV\rangle_s=c\langle VW\rangle s\exp(\tfrac{1}{2}c^2\langle V^2\rangle_s). \tag{A2.2}$$

Here we shall show that this is the case for the linear functions V and W of the Gaussian noise $\zeta(x)$, which satisfies

$$\langle\zeta(x_1)\ldots\zeta(x_n)\rangle_s=\begin{cases}0 & n\text{ odd}\\ \Sigma\Pi\langle\zeta(x_i)\zeta(x_j)\rangle_s & n\text{ even}\end{cases} \tag{A2.3}$$

$\Sigma\Pi$ means that we multiply $n/2$ products $\langle\zeta(x_i)\zeta(x_j)\rangle_s$ and sum over the $(n-1)!!$ different combinations (i,j). x_i can stand for either t_i, which is needed in section 4, or (x_i,t_j), which we use in section 5.

We begin with the proof of (A2.1). Expanding the exponential, we have

$$\langle e^{cV}\rangle_s=\left\langle\sum_{k=0}^{\infty}\frac{c^k}{k!}V^k\right\rangle_s=\sum_{k=0}^{\infty}\frac{c^k}{k!}\langle V^k\rangle_s. \tag{A2.4}$$

Since V is assumed to be linear in the noise, we can use the Gaussian behaviour of the noise to compute

$$\langle V^k(x)\rangle_s = \begin{cases} 0 & k \text{ odd} \\ \langle V^2(x)\rangle_s^l (2l-1)!! & k = 2l \text{ even.} \end{cases} \tag{A2.5}$$

Here we have taken the limit $x_i \to x$ and used the fact that there are $(k-1)!!$ terms in the sum in (A2.3).

Inserting this into equation (A2.4), we have

$$\begin{aligned} \langle e^{cV}\rangle &= \sum_{k=0}^{\infty} \frac{c^k}{k!} \langle V^2\rangle_s^l (2l-1)!! \qquad k \text{ even} \\ &= \sum_{l=0}^{\infty} \frac{c^{2l}}{(2l)!} \langle V^2\rangle_s^l (2l-1)!! \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{c^2\langle V^2\rangle_s}{2}\right)^l \\ &= \exp(\tfrac{1}{2}c^2\langle V^2\rangle_s). \end{aligned} \tag{A2.6}$$

Thus, we have proven equation (A2.1).

Equation (A2.2) can be proven in the same manner. Expanding the exponential, we have

$$\langle W e^{cV}\rangle_s = \left\langle \sum_{k=0}^{\infty} \frac{c^k}{k!} W V^k \right\rangle_s = \sum_{k=0}^{\infty} \frac{c^k}{k!} \langle W V^k\rangle_s. \tag{A2.7}$$

Since both W and V are linear in ζ, we use (A2.3) to obtain

$$\langle W V^k\rangle_s = \begin{cases} 0 & k \text{ even} \\ k\langle VW\rangle_s \langle V^{k-1}\rangle_s & k \text{ odd.} \end{cases} \tag{A2.8}$$

The factor of k occurs because there are k possible choices for V, which can be paired with W. For $k = 2l+1$ we have

$$\begin{aligned} k\langle VW\rangle_s \langle V^{k-1}\rangle_s &= (2l-1)\langle VW\rangle_s \langle V^{2l}\rangle_s \\ &= (2l+1)!!\langle VW\rangle_s \langle V^2\rangle_s^l. \end{aligned} \tag{A2.9}$$

Therefore, (A2.7) evaluates to

$$\begin{aligned} \langle W e^{cV}\rangle_s &= \sum_{l=0}^{\infty} \frac{c^{2l+1}}{(2l+1)!} (2l+1)!!\langle VW\rangle_s \langle V^2\rangle_s^l \\ &= \sum_{l=0}^{\infty} \frac{c^{2l-1}}{(2l)!} \langle WV\rangle_s \langle V^2\rangle_s^l \\ &= c\langle VW\rangle_s \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{c}{2}\langle V^2\rangle_s\right)^l \\ &= c\langle VW\rangle_s \exp(\tfrac{1}{2}c^2\langle V^2\rangle_s). \end{aligned} \tag{A2.10}$$

Appendix 3. Computation of the ensemble averages $\langle QW\rangle_s$ and $\langle Q^2\rangle_s$

For our general stochastic perturbation term, given by $\zeta(x, t)R[u]$, we needed the ensemble averages $\langle QW\rangle_s$ and $\langle Q^2\rangle_s$ in order to discuss the average behaviour of the fluctuating soliton. We found Q and W in equations (114) and (115) as

$$W=\eta_0^{2p}\int_0^t \langle\hat{R}\zeta|\mathrm{sech}^2\,\phi\rangle\exp[\tfrac{4}{3}\varepsilon\gamma(s-t)]\,\mathrm{d}s \tag{A3.1}$$

$$Q=-\frac{\eta_0^{p-3}}{4}\int_0^t \mathrm{d}s\langle\hat{R}\zeta|\phi\,\mathrm{sech}^2\,\phi+\tanh\phi+\tanh^2\phi\rangle-2\int_0^t \mathrm{d}s\,W(s)+\frac{1}{4\eta_0^2}\chi_0 W(t). \tag{A3.2}$$

As each of these is linear in ζ, we see that the products QW and Q^2 are quadratic in the noise. In order to average over these products, we define

$$\langle\zeta(x,t)\zeta(x',t')\rangle_s=N\delta(t-t')\delta(x-x'). \tag{A3.3}$$

There are no complications in integrating out the $\delta(x-x')$. In doing so we are left with three types of integrals over the phase. We define these as

$$G_1\equiv\int_{-\infty}^{\infty}\mathrm{d}\phi\,\hat{R}^2[u_0](\phi\,\mathrm{sech}^4\,\phi+\mathrm{sech}^2\,\phi\tanh\phi+\mathrm{sech}^2\,\phi-\mathrm{sech}^4\,\phi) \tag{A3.4}$$

$$G_2\equiv\int_{-\infty}^{\infty}\mathrm{d}\phi\,\hat{R}^2[u_0]\,\mathrm{sech}^4\,\phi \tag{A3.5}$$

$$G_3\equiv\int_{-\infty}^{\infty}\mathrm{d}\phi\,\hat{R}^2[u_0](\phi\,\mathrm{sech}^2\,\phi+\tanh\phi+\tanh^2\phi)^2. \tag{A3.6}$$

We are now left with integrations over the time variables. We can write the averages as

$$\langle QW\rangle_s=-\tfrac{1}{4}N\eta_0^{2p-3}(G_1V_1+8\eta_0^3G_2V_2-\eta_0\chi_0G_1V_3) \tag{A3.7}$$

$$\langle Q^2\rangle_s=\tfrac{1}{16}N\eta_0^{2p-6}(G_3V_4+64\eta_0^6G_2V_5+\eta_0^2\chi_0^2G_2V_3+16\eta_0^3G_1V_6-2\eta_0\chi_0G_1V_1-16\eta_0^4\chi_0G_2V_2) \tag{A3.8}$$

where we have defined the time integrals by

$$V_1\equiv\int_0^t \mathrm{d}s\exp[\tfrac{4}{3}\varepsilon\gamma(s-t)]\int_0^t \mathrm{d}s'\delta(s-s') \tag{A3.9}$$

$$V_2\equiv\int_0^t \mathrm{d}s\exp[\tfrac{4}{3}\varepsilon\gamma(s-t)]\int_0^t \mathrm{d}s'\int_0^{s'}\mathrm{d}s''\exp[\tfrac{4}{3}\varepsilon\gamma(s''-s')]\delta(s''-s) \tag{A3.10}$$

$$V_3\equiv\int_0^t \mathrm{d}s\exp[\tfrac{4}{3}\varepsilon\gamma(s-t)]\int_0^t \mathrm{d}s'\exp[\tfrac{4}{3}\varepsilon\gamma(s'-t)]\delta(s'-s) \tag{A3.11}$$

$$V_4\equiv\int_0^t \mathrm{d}s\int_0^t \mathrm{d}s'\delta(s-s') \tag{A3.12}$$

$$V_5\equiv\int_0^t \mathrm{d}s\int_0^s \mathrm{d}s_1\exp[\tfrac{4}{3}\varepsilon\gamma(s_1-s)]\int_0^t \mathrm{d}s'\int_0^{s'}\mathrm{d}s_2\exp[\tfrac{4}{3}\varepsilon\gamma(s_2-s')]\delta(s_1-s_2) \tag{A3.13}$$

$$V_6\equiv\int_0^t \mathrm{d}s\int_0^t \mathrm{d}s'\int_0^{s'}\mathrm{d}s_1\exp[\tfrac{4}{3}\varepsilon\gamma(s_1-s')]\delta(s_1-s). \tag{A3.14}$$

These integrals are computed separately, noting that

$$\int_0^{t_1} ds_1 \int_0^{t_2} ds_2 f(s_2)\delta(s_1-s_2) = \int_0^{\min(t_1,t_2)} dt f(t). \tag{A3.15}$$

We have

$$V_1 = \int_0^t ds \exp[\tfrac{4}{3}\varepsilon\gamma(s-t)] = \frac{3}{4\varepsilon\gamma}[1-\exp(\tfrac{8}{3}\varepsilon\gamma t)] \tag{A3.16}$$

$$\begin{aligned} V_2 &= \int_0^t ds' \int_0^{s'} ds'' \exp[\tfrac{4}{3}\varepsilon\gamma(2s''-t-s')] \\ &= \frac{3}{8\varepsilon\gamma}\int_0^t ds' \exp[-\tfrac{4}{3}\varepsilon\gamma(s'+t)][\exp(\tfrac{8}{3}\varepsilon\gamma s')-1] \\ &= \frac{9}{32\varepsilon^2\gamma^2}[1+\exp(-\tfrac{8}{3}\varepsilon\gamma t)-2\exp(-\tfrac{4}{3}\varepsilon\gamma t)] \end{aligned} \tag{A3.17}$$

$$V_3 = \int_0^t ds \exp[\tfrac{4}{3}\varepsilon\gamma(2s-2t)] = \frac{3}{8\varepsilon\gamma}[1-\exp(-\tfrac{8}{3}\varepsilon\gamma t)] \tag{A3.18}$$

$$V_4 = \int_0^t ds \int_0^t ds'\delta(s-s') = t \tag{A3.19}$$

$$\begin{aligned} V_5 &= \int_0^t ds \int_0^t ds' \int_0^{\min(s,s')} dr \exp[\tfrac{4}{3}\varepsilon\gamma(2r-s-s')] \\ &= \frac{3}{8\varepsilon\gamma}\int_0^t ds\bigg(\int_0^s ds' \exp[-\tfrac{4}{3}\varepsilon\gamma(s+s')][\exp(\tfrac{8}{3}\varepsilon\gamma s')-1] \\ &\quad + \int_s^t ds' \exp[-\tfrac{4}{3}\varepsilon\gamma(s+s')][\exp(\tfrac{8}{3}\varepsilon\gamma t)-1]\bigg) \\ &= \frac{9}{32\varepsilon^2\gamma^2}\int_0^t ds\{2-2\exp(-\tfrac{4}{3}\varepsilon\gamma s)-\exp[\tfrac{4}{3}\varepsilon\gamma(s-t)]+\exp[-\tfrac{4}{3}\varepsilon\gamma(s+t)]\} \\ &= \frac{27}{128\varepsilon^3\gamma^3}\left(\frac{8\varepsilon\gamma t}{3}+4\exp(-\tfrac{4}{3}\varepsilon\gamma t)-1-\exp(\tfrac{8}{3}\varepsilon\gamma t)\right) \end{aligned} \tag{A3.20}$$

$$\begin{aligned} V_6 &= \int_0^t ds \int_0^s ds' \exp[\tfrac{4}{3}\varepsilon\gamma(s'-s)] \\ &= \frac{3}{4\varepsilon\gamma}\int_0^t ds[1-\exp(-\tfrac{4}{3}\varepsilon\gamma s)] \\ &= \frac{9}{16\varepsilon^2\gamma^2}\left(\frac{4\varepsilon\gamma}{3}t+\exp(-\tfrac{4}{3}\varepsilon\gamma t)-1\right). \end{aligned} \tag{A3.21}$$

Inserting these results into equations (A3.7) and (A3.8), we find the averages sought as

$$\begin{aligned} \langle QW\rangle_s &= -\frac{3N\eta_0^{2p-3}}{16\varepsilon\gamma}G_1[1-\exp(-\tfrac{8}{3}\varepsilon\gamma t)]+\frac{3\eta_0^{2p-2}N}{32\varepsilon\gamma}\chi_0 G_2[1-\exp(-\tfrac{8}{3}\varepsilon\gamma t)] \\ &\quad -\frac{9N\eta_0^{2p}}{16\varepsilon^2\gamma^2}G_2[1+\exp(-\tfrac{8}{3}\varepsilon\gamma t)-2\exp(-\tfrac{4}{3}\varepsilon\gamma t)] \end{aligned} \tag{A3.22}$$

and

$$\langle Q^2\rangle_s=\frac{\eta_0^{2p-6}}{16}NG_3t+\frac{27\eta_0^{2p}}{32\varepsilon^2\gamma^2}NG_2\left(\frac{8\varepsilon\gamma t}{3}+4\exp(-\tfrac{4}{3}\varepsilon\gamma t)-1-\exp(-\tfrac{8}{3}\varepsilon\gamma t)\right)$$
$$+\frac{3N\eta_0^{2p-4}}{128\varepsilon\gamma}\chi_0^2G_2[1-\exp(-\tfrac{8}{3}\varepsilon\gamma t)]$$
$$+\frac{9N\eta_0^{2p-3}}{16\varepsilon^2\gamma^2}G_1\left(\frac{4\varepsilon\gamma t}{3}+\exp(-\tfrac{4}{3}\varepsilon\gamma t)-1\right)$$
$$-\frac{3\eta_0^{2p-5}}{32\varepsilon\gamma}\chi_0G_1[1-\exp(-\tfrac{4}{3}\varepsilon\gamma)]$$
$$-\frac{9N\eta_0^{2p-2}}{32\varepsilon^2\gamma^2}G_2[1+\exp(-\tfrac{8}{3}\varepsilon\gamma t)-2\exp(-\tfrac{4}{3}\varepsilon\gamma t)]. \quad \text{(A3.23)}$$

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